The 20 matchstick triangle challenge; an activity to foster reasoning and problem solving

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Introduction

In this article we look at a simple geometry problem that also involves some reasoning about number combinations, and show how it was used in a Year 7 classroom. The problem is accessible to students with a wide range of abilities, and provides scope for stimulating extensive discussion and reasoning in the classroom, as well as an opportunity for students to think about how to work systematically. Pat, the first author and a classroom teacher, used the problem with her students and we will present some of the strategies, solutions, and issues that they encountered and discussed. Helen, the second author who works with pre-service and in-service teachers, has used this problem with teachers and likes thinking about tasks that are good for fostering reasoning and problem solving.

Pat first encountered this problem during Helen's presentation at a local mathematics teachers' conference (and, unfortunately, Helen cannot remember where she first came across it). The wording of the original problem was to find as many triangles as possible with a perimeter of 20 cm, where the side lengths have to be whole numbers. When Pat decided to use the activity with her Year 7 class she had the students working in pairs and adapted the task to make it more hands-on. She called the task the *"Triangle Challenge"* to appeal to the students' competitive spirit and restated the problem in terms of building triangles out of matchsticks:

> Make all of the possible triangles that can be made from 20 matchsticks. You must use all 20 matchsticks for each triangle. You must record which triangles you have made in some way. How will you know when you have all the possible triangles?

Pat's students took to the task with gusto. It was not long before students were asking, "Can we break the matchsticks?" She gave a follow-up instruction that no matchsticks could be broken and there were to be no gaps between the matchsticks. Some students had difficulties with the construction of the triangles, especially with ensuring the matchsticks were touching and that the sides were straight. Some students used rulers to help with straightening the sides, so this technique was shared with the

whole class. Helen wonders if this awkwardness with the materials may actually help students bridge the concrete and abstract characteristics of the shapes, since the students have to start thinking about whether or not the edges will really join up even though it is not entirely clear that they will because of the practical limitations of the matchsticks.

Observed student approaches

When Helen had first posed the problem, she had only seen her own solution (although she was confident the problem would be a good one for students), and so was curious as to what strategies that students might use when tackling the problem. Pat gave her students no hints at all as to how the triangles should be recorded, and told them to choose any method that suited them. A summary of how students tackled the challenge and recorded their triangles is presented in Table 1. There were a total of 24 students, working in pairs, on this particular day.

Table 1: Students' approaches to the 20 matchstick triangle challenge.

Approach taken	Number of pairs (out of 12) who demonstrated this approach
Recorded triangles by drawing to scale* (e.g., Figure 1)	2
Recorded triangles by sketching matchsticks* (e.g., Figure 2)	2
Recorded triangles by worded description (e.g., 2 across, 9 up, 9 up*; Figure 2)	4
Recorded triangles with labelled sides, with the triangles not drawn to scale (e.g., Figure 3)	5
Did not draw triangles (e.g., used description only or a graphical record; Figures 4 and 5)	2
Worked in a systematic manner (e.g., increasing one side in turn by a fixed amount; Figure 5 partly)	2
Started with isosceles triangles (e.g., 2, 9, 9; Figures 3b and 5)	6

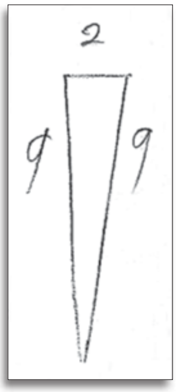
* students may have used this approach as well as another

After about 40 minutes, Pat stopped the triangle construction and recording stage and started a class discussion. The discussion also continued into the following day's maths lesson.

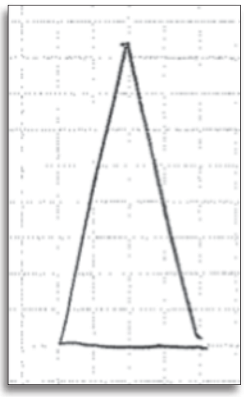
Class discussion

Helen and Pat both believe that class discussion provides an opportunity to make explicit and public the reasoning with which students have engaged in their problem solving process, and to stimulate further reasoning, conjecturing, hypothesising, and refutation. Pat started by asking questions about the actual construction of the triangles. Most students had enjoyed the construction process, but said it was difficult at times to keep the sides straight, while ensuring the matchsticks were touching and that the whole triangle was neat and complete.

Pat then turned to specific questions about how students recorded their triangles. Two groups of students attempted to draw the triangles to scale, as shown in Figure 1.



Scale diagram with 5 mm = 1 matchstick



Use of grid paper to record a scale diagram. 1 grid box = 1 matchstick

Figure 1: Two approaches to recording the triangles with scale diagrams.

Two other groups sketched in the matchsticks, without any attempt at drawing to scale, as seen in Figure 2. Interestingly, the students drew 'heads' on the matchsticks, even though they used coloured sticks without heads.

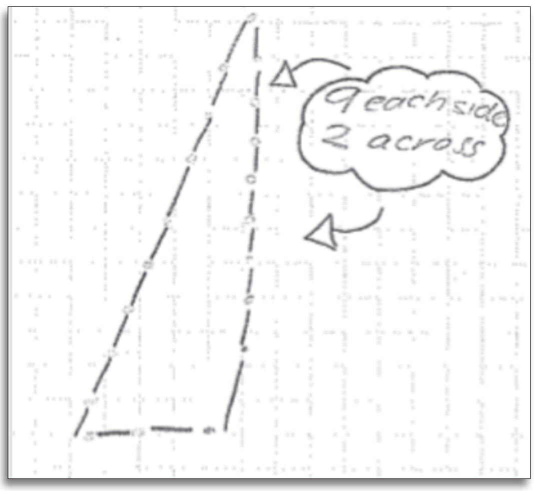


Figure 2: Triangles recorded with no use of scale.

Five groups drew triangles with labelled sides, as seen in Figures 3a and 3b. In Figure 3a, the students attempted to represent aspects of each triangle's shape (e.g., in their second diagram, it is obvious that it is an isosceles triangle), whereas the pair responsible for Figure 3b have made no size and shape distinctions for the collection of isosceles triangles. When Pat asked them about this, they said it was not necessary, that recording the side lengths was all that was needed to show which triangles were possible. Here we can see the abstraction of properties from the concrete materials, with the students realising that the side lengths are sufficient to distinguish among the different possible triangles.

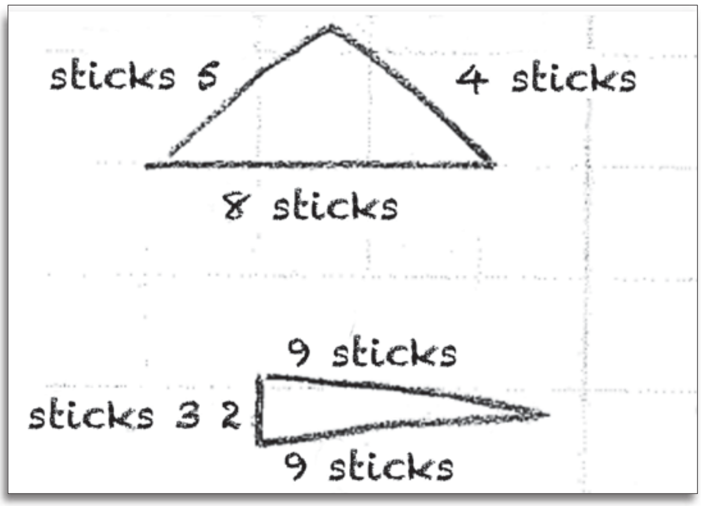


Figure 3a: Triangles with recorded sides and some indication of the variation in shape.

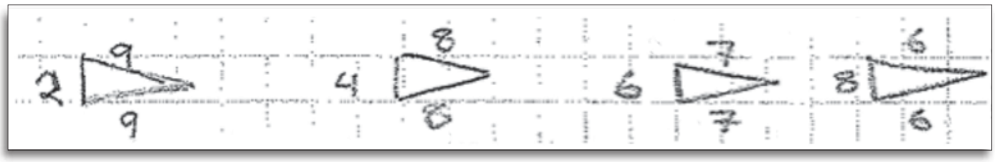


Figure 3b: Triangles with recorded sides, with no indication of shape variation.

There were only two groups that did not draw the triangles which they created. In Figure 4, we can see that one group focused on recording one side length as the base, and the remaining sides as going 'up' from there. It is likely that, because of this focus on the base length, this group did not see that the first two triangles made and recorded were identical.

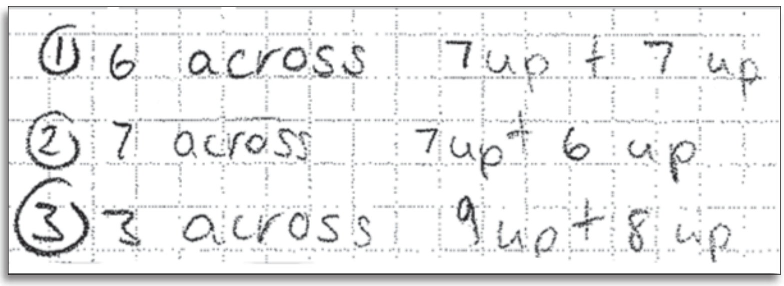


Figure 4: Recording the triangles without diagrams, using base length as point of reference.

The work of the group shown in Figure 5 is particularly interesting: they used a bar-graph-like representation of the side lengths, with three 'bars' showing the lengths of the sides. When Pat asked why they chose to represent their triangles in this form, they answered that it was to ensure that they did not repeat any triangles. In fact their third and sixth triangles are actually duplicates (with sides of 6, 7 and 7 matchsticks). It is worth thinking about how the recording method might be modified to avoid duplicates. This group showed a systematic approach, with the 'base' increasing by two matchsticks initially. They also started with isosceles triangles, which was a popular approach in half of all the groups. When Helen conducted this activity with some teachers, she found that they too, generally started with isosceles triangles. A useful question to raise with the problem solvers at this point is to ask whether or not it is possible to have an isosceles triangle with an odd number of matchsticks as its base length, and, if not, why.

Pat was pleased with the variety of approaches taken constructing and recording the triangles. It was, however, interesting to note that the student pairs varied in the degree to which they had a systematic way of listing the possibilities. This will be addressed later.

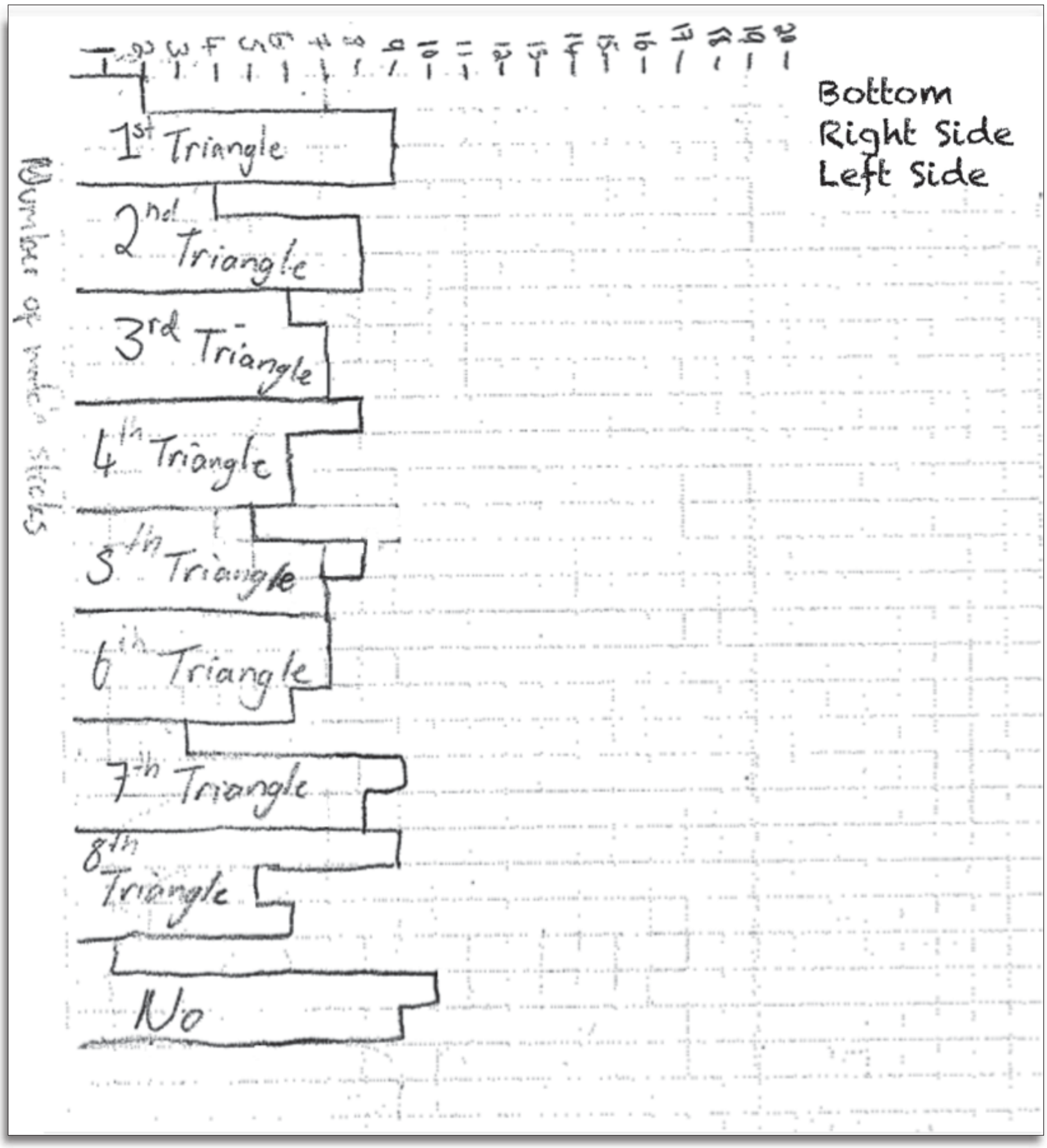


Figure 5: Recording the side lengths using a graphical approach.

Pat then asked the class if anyone made a triangle with sides of 1, 1 and 18 matchsticks. They laughed and told her it would be impossible. They discussed why this was so and represented the matchsticks with scaled diagrams on the board. It was obvious that the two sides of length 1, when attached to each end of a base of 18, could not touch to make a triangle. She then asked the students how long the longest side of a 20-matchstick triangle could be. Several groups had made triangles of side lengths 1, 9 and 10, or 5, 5 and 10. No one claimed to make a triangle with anything longer than 10 matchsticks as the longest side.

So the discussion turned to whether or not it was possible to make a triangle with a side of 10 matchsticks. Pat drew scaled diagrams on the board of different sized triangles. It was obvious to the students that 11 on the base would be too long, as the other two sides would not be able to touch. But with 10 as a base, there was still confusion. It took several diagrams as well as viewing a simulation (*Math Warehouse*, n.d.; another useful resource is *Hotmath*, n.d.) for the students to understand that two sides added together need to be longer than the third side for a triangle to be made. After further class discussion, with lots of examples, many of the students worked out that they really only needed to check that the sum of the two shorter sides of a triangle was longer than the other side. They did not need to check all three combinations of sides.

Following this exploration, the students wanted to know why they were able to physically make some triangles with one side length of 10 matchsticks. This led to a discussion about the process of making the triangles. They noted that the matchsticks might not have been exactly the same length, that there may have been gaps or slight overlaps when constructing the triangles, that the sides might not have been completely straight, or that the sides may not have touched at the vertices.

When Helen conducted this activity with some teachers, she raised the question of whether or not we should regard three-sided shapes such as those with sides 1, 9, 10 and 5, 5, 10 as triangles. Pat's students had felt uncertain about these shapes, and Helen's teachers felt a bit uncomfortable with them too, in part because the materials were misleading, but also because, in some sense, the edges do join up, although the resulting shape looks like a line segment. When Helen asked the teachers for the definition of a triangle they said that a triangle is a shape that comprises three sides that join up, a definition likely to be given by students as well. When asked whether or not a 1, 9, 10 shape fits this definition Helen's teachers were forced to concede that it does, but then they wanted to refine their definition of triangle (usually beyond the definition that they have used all their lives up to this point!). Helen encouraged them to stay with the "three joined-up straight sides" definition, and then to think about the 1, 9, 10 shape a little more. She asked further questions about this shape, such as whether or not it satisfies "the sum of the angles is 180°" (it does), and got them to work out its area using $\frac{1}{2} \times$ base $\times$ height, which actually yields an appropriate answer of 0 (and, for more advanced students it is possible to explore the $\frac{1}{2} \times a \times b \times \sin C$ formula for area as well). What is intriguing, in conclusion, is that the 1, 9, 10 shape does not yield any surprising contradictory results if we think about it as a triangle and consider triangle properties.

In fact, the only tricky part concerns the sum of the sides. As Pat's class discovered, if you want to have a triangle that does not end up looking just like a straight line segment (as the 1, 9, 10 triangle does) then you need the sides to satisfy the property that the sum of any two sides is greater than the length of the third side. This is known as the 'triangle inequality'.

However, the most general mathematical version of the triangle inequality states that the sum of the two sides need only be greater than or equal to the third, and thus allows the 1, 9, 10 shape to be included as a triangle. We might call such triangles 'degenerate triangles': they are triangles, but taken to extremes and with extreme properties! Helen believes that all of these issues can be discussed with students, leading to a good discussion of how to properly define shapes, and the important role of definitions in mathematics more generally. It also gives students an opportunity to explore the implications of these definitions on the properties of these objects.

The final discussion in Pat's class was about whether or not the students had found all of the possible triangles. She mentioned the words "working systematically". None of the students understood what she meant, but when she gave student examples of attempting to work in some sort of order, several of the students said they did try to do that. She then showed them how to draw a table to record the results. It did not matter which was the first, second or third side, because the triangles could be rotated or flipped. This is another really powerful discussion to have with students, leading to the conclusion that if you want to be systematic you can just record the sides in order from smallest to biggest. This would help in locating duplicates. Pat's class developed a table together on the board (see Table 2). They started with the shortest possible values for the base, and began with isosceles triangles where possible, then decreasing the second side by 1 while increasing the third side by 1 each time. Students realised quickly that once one side reached 10, they needed to start the process again with a new base size.

The students easily understood that once they reached 6, 6, 8, the next one would be 6, 5, 9, which was already there (as 5, 6, 9). This meant they had reached the end of the table, and any other possibilities would just be duplicates. They were surprised that there were only eight triangles possible with 20 matchsticks. The students who drew the graphical representations of the possible side lengths (Figure 5) managed to make and record seven of the eight possible triangles. Another group managed to record six triangles and four groups recorded five triangles.

Table 2. The list of all possible 20-matchstick triangles. ('Degenerate triangles' are struck through.)

Side 1 ("base")	Side 2	Side 3
~~1~~	~~9~~	~~10~~
2	9	9
~~2~~	~~8~~	~~10~~
3	8	9
~~3~~	~~7~~	~~10~~
4	8	8
4	7	9
~~4~~	~~6~~	~~10~~
5	7	8
5	6	9
~~5~~	~~5~~	~~10~~
6	7	7
6	6	8

The fact that there are only eight possible triangles using 20 matchsticks intrigued Helen in the process of preparing this article. She started to explore an extension problem that could be presented easily to students who had followed Pat's suggestion of exploring other values for the total number of matchsticks being used. Helen's extension problem asks you to imagine that you are trying to make triangles out of matchsticks, and they have to have a whole number of matchsticks on each side. You are no longer restricted to having 20 matches in total; you can have as many or as few as you like. Can you work out what triangles are possible? Can you characterise all of them? Written more succinctly:

> Find and describe *all* the triangles that can be made with whole number side lengths.

Answering this fully is likely to be a significant challenge for younger high school classes, but should be accessible to them if they have done the initial work on the 20 matchstick problem and perhaps a few other fixed values, and have used these examples to develop a sound understanding of the triangle inequality relationship.

Common errors

Returning to the work of Pat's class on the original 20 matchsticks problem, there were some underlying reasons for the fact that students initially could not list all the possibilities. Several pairs of students made duplicate triangles, which were just rotations of others previously made. Another common error observed was the recorded triangles did not have three sides adding up to 20 matchsticks. This may have been because not all the matchsticks were used, or just that they were recorded incorrectly. The groups which attempted to draw triangles to scale did make errors in measuring; in particular, the group which used one square grid equalling one matchstick assumed that the diagonal of the square was equal to the length of the square. This is not totally surprising as these students had had no exposure to Pythagoras' Theorem at this stage.

Conclusion

Given more time, Pat would have liked to explore working with a different number of matchsticks. Would the students attempt to work systematically? How would they record their results? It would also have been interesting to extend some students with an introduction to Pythagoras' Theorem.

It should be noted that, several weeks later, the students in Pat's class sat a triangles test. One of the questions was "Can you make a triangle with sides 3 cm, 6 cm and 10 cm? Explain your answer." Of the 26 students in the class, 18 of them were able to say that this triangle would be impossible, and give a reasonable explanation as to why.

It is interesting to note that the triangle inequality is not mentioned in the *Australian Curriculum: Mathematics* (ACARA, 2014). This is a shame, because it is applied frequently, even if most people are doing so only instinctively: whenever someone walks diagonally across a rectangular or quadrilateral-shaped grassed area instead of walking around they are using the fact that the diagonal will be shorter than the sum of the two sides that they are avoiding. The triangle inequality also acts as a nice

check for solutions to cosine rule problems that require the finding of the third side of a triangle given two sides and an angle: when the third side is calculated it must be shorter than the sum of the other two given sides (which of course is also true in the specific case of right-angled triangles, where the hypotenuse is shorter than the sum of the legs). Groth (2005) suggests another activity using spaghetti that allows students to build understanding of this key theorem.

The 20 matchstick problem—which is readily stated, and which can be tackled with physical manipulatives—provides easy access to the triangle inequality, with the advantage that students are able to discover its principles for themselves and get a feel for why it must be true. It is also a rich problem-solving task. The significant reasoning that must be produced in order to check which triangles are valid, and to enumerate all the possibilities, is within reach of young high school students, as Pat's class has shown. This problem provides students with the valuable opportunity to undertake initial exploration to understand the situation and then progress to working systematically, with careful argument and justification, to ensure that all cases are considered. Such activities will build students' problem solving and reasoning skills, as key proficiencies.

References

Australian Curriculum, Assessment and Reporting Authority [ACARA]. (2014). *The Australian Curriculum: Mathematics*. Retrieved from http://www.australiancurriculum.edu.au/Mathematics/curriculum/f-10?layout=1

Groth, R. E. (2005). Linking theory and practice in teaching geometry. *Mathematics Teacher, 99*(1), 27–30.

Hotmath. (n.d.). *Triangle inequalities*. Retrieved from http://hotmath.com/learning_activities/interactivities/triangleInequality.swf

Math Warehouse. (n.d.). *Triangle inequality theorem rule explained*. Retrieved from http://www.mathwarehouse.com/geometry/triangles/triangle-inequality-theorem-rule-explained.php